JOURNAL OF COMBINATORIAL THEORY, Series (B) 28, 180–183 (1980)

Minimal Separating Sets of Maximum Size

Y. ALAVI*

Western Michigan University, Kalamazoo, Michigan 49008,

M. BEHZAD

University of Northern Iran

AND

E. A. NORDHAUS

Michigan State University, East Lansing, Michigan 48824

Communicated by the Editors

Received February 17, 1976

A connected graph G can be disconnected or reduced to a single vertex by removing an appropriate subset of the vertex set $V(G)$, and can be disconnected by removing a suitable subset of the edge set $E(G)$. Attention has usually been centered on separating sets having minimum cardinality, and parameters called the vertex connectivity and the edge connectivity defined. These classical concepts are generalized by using separating sets which are minimal. By considering the maximum as well as the minimum cardinality of such sets, one defines vertex and edge connectivity parameters. Sharp upper bounds are established for these numbers and their values computed for certain classes of graphs. An analogue of Whitney's theorem on connectivity is obtained. Parameters are also defined for minimal separating sets consisting of a mixture of vertices and edges, and these are shown to depend on the maximum and minimum values of the vertex and edge connectivity parameters.

1. PRELIMINARIES

Let G be a connected undirected graph with no loops or multiple edges having vertex set $V(G)$ and edge set $E(G)$, with $|V(G)| = p > 1$ and $|E(G)| = q$, so that $q \geqslant p - 1$. A set S of elements of $V(G) \cup E(G)$ whose removal from G results in a disconnected graph or a single vertex is called a

* Research supported in part by Western Michigan University (Faculty Research Fellowship).

0095-8956/80/020180–04$02.00/0

separating set for G, and will be assumed to be minimal. One observes that the complete graph K_p cannot be disconnected by removing vertices, but can be reduced to a single vertex by removing any $p - 1$ vertices.

A minimal separating set S_0 whose elements are vertices of G is called a *minimal vertex separating set*, and the minimum and the maximum cardinalities of such sets are denoted by $\kappa_0(G)$ and $\kappa_0'(G)$, respectively, so $1 \leqslant \kappa_0(G) \leqslant |S_0| \leqslant \kappa_0'(G) \leqslant p - 1$. Similarly a minimal separating set S_1 whose elements are edges of G is called a *minimal edge separating set*, with minimum and maximum cardinalities denoted by $\kappa_1(G)$ and $\kappa_1'(G)$ respectively, where $1 \leqslant \kappa_1(G) \leqslant |S_1| \leqq \kappa_1'(G) \leqq q$. It is obvious that $\kappa_0(G) = 1$ if and only if G has a cut vertex and that $\kappa_1(G) = 1$ if and only if G has a bridge. In the literature, $\kappa_0(G)$ is called the vertex (or point) connectivity and $\kappa_1(G)$ the edge (or line) connectivity.

The minimum degree of any vertex of G will be denoted by $\delta(G)$ and the maximum degree by $\Delta(G)$. It follows from the definitions and a result of Whitney [6], that $1 \leqslant \kappa_0(G) \leqslant \kappa_1(G) \leqq \delta(G) \leqq \Delta(G) \leqq p - 1$. It will be shown in Theorem 2 that an analogue of Whitney's theorem, namely $\kappa_0'(G) \leqq \kappa_1'(G)$, holds when $\Delta(G) < p - 1$.

2. Edge Connectivity

We next develop two sharp upper bounds for the edge connectivity parameter $\kappa_1'(G)$. The symbols $[x]$ and $\{x\}$ will denote respectively the greatest integer $\leqq x$ and the least integer $\geqslant x$.

THEOREM 1. *Let G be a connected graph of order p having q edges. Then* $\kappa_1'(G) \leqq \min(q - p + 2, [p^2/4])$.

Proof. Let S be any minimal separating edge set for G. Then $G - S$ has exactly two components, G_1 and G_2, with say p_1 and p_2 vertices, $p_1 + p_2 = p$. The number of edges in G_i must be at least $p_i - 1$ so that $|S| \leqq q - p + 2$. On the other hand, each edge of S must join a vertex of G_1 to one of G_2 so that $|S| \leqq p_1 p_2$. Now $p_1 p_2$ is a maximum when p_1 and p_2 are as nearly equal as possible, and then $p_1 p_2 = [p/2] \cdot \{p/2\} = [p^2/4]$. This proves the theorem. The bounds stated are sharp, since $\kappa_1'(K_{m,n}) = 1 + (m - 1)(n - 1) = q - p + 2$ and $\kappa_1'(K_p) = [p^2/4]$.

If a connected graph G has an orientable imbedding on a surface of genus $\gamma(G)$, then by the extended Euler polyhedral formula, $F(G) - q + p = 2 - 2\gamma(G)$, where $F(G)$ is the number of faces in the imbedding. By Theorem 1, $\kappa_1'(G) \leqq F(G) + 2\gamma(G)$. In particular if G is planar, $\gamma(G) = 0$ and $\kappa_1'(G) \leqq F(G)$. Equality can occur, as is the case for the cartesian product $G = P_m \times P_n$ of two disjoint paths P_m and P_n.

3. Vertex Connectivity

We next consider the vertex connectivity parameters $\kappa_0(G)$ and $\kappa_0'(G)$. The set of all vertices adjacent to a given vertex of G contains a minimal vertex separating set, so that, as noted earlier, $\kappa_0(G) \leqq \delta(G)$.

If S_0 is any minimal vertex separating set for G, the graph $G - S_0$ is either disconnected or consists of a single vertex. One can readily show that $\kappa_0'(G)$ attains its maximum value $p - 1$ if and only if G has a vertex of degree $p - 1$. It also follows that $\kappa_0(G) = \kappa_0'(G) = p - 1$ only for the complete graph K_p, and that the complete graph is the only connected graph which cannot be disconnected by removing vertices.

The next result relates $\kappa_0'(G)$ and $\kappa_1'(G)$, and provides an analogue of Whitney's theorem on connectivity.

THEOREM 2. *Let G be a connected graph of order $p \geqslant 4$ having $\Delta(G) < p - 1$. Then $\kappa_0'(G) \leqq \kappa_1'(G)$.*

Proof. Suppose that $\kappa_1'(G) < \kappa_0'(G)$. Since G has no vertex of degree $p - 1$, then $\kappa_0'(G) \leqq p - 2$ and $\kappa_1'(G) \leqq p - 3$. Let S_0 be a minimal vertex separating set for G of maximum size, so $|S_0| = \kappa_0'(G)$. Then $G - S_0$ has at least two vertices and k components $G_1, G_2, ..., G_k$ where $k \geqslant 2$. Every vertex of S_0 is joined by at least one edge to each of these k components, since otherwise the minimality of S_0 is contradicted. Let S_1 be the set of all edges of G joining vertices of S_0 to the vertices of the single component G_1. This set is a minimal edge separating set with cardinality at least $\kappa_0'(G)$, so $|S_1| > \kappa_1'(G)$, a contradiction.

As a consequence of Theorem 2, the upper bound $q - p + 2$ found for the parameter $\kappa_1'(G)$ in Theorem 1 is also an upper bound for $\kappa_0'(G)$ when $\Delta(G) < p - 1$. This bound is sharp, as shown by the graph $G = K_{2,n}$ with $n \geqslant 2$. If $\Delta(G) = p - 1$, then $\kappa_0'(G) = p - 1$ and can exceed $q - p + 2$, as shown by the graph $K_{1,n}$ with $n > 1$. However, one can also have $q - p + 2 > p - 1$, as shown by the graph K_p with $p \geqslant 4$. The analogue of Whitney's theorem on connectivity therefore fails for certain connected graphs with $\Delta(G) = p - 1$. Such graphs have exactly one vertex v of degree $p - 1$ and have v as a cut vertex, as is readily demonstrated.

4. Mixed Connectivity Numbers

A minimal separating set S whose elements belong to $V(G) \cup E(G)$ may be called a *mixed separating set*, and the minimum and maximum cardinalities of such sets denoted by $\kappa_2(G)$ and $\kappa_2'(G)$ respectively. It is immediate that $\kappa_2(G) \leqslant \kappa_0(G)$ and $\kappa_2'(G) \geqslant \max(\kappa_0'(G), \kappa_1'(G))$. However, we show that equality holds in both cases.

THEOREM 3. *Let G be a connected graph of order p with maximum degree* $\Delta(G)$. *Then* (a) $\kappa_2(G) = \kappa_0(G)$, (b) $\kappa_2'(G) = \kappa_1'(G)$ *if* $\Delta(G) < p - 1$, *and* (c) $\kappa_2'(G) = \max(p - 1, \kappa_1'(G))$ *if* $\Delta(G) = p - 1$.

Proof. Let S be a minimal mixed separating set for graph G having m vertices and n edges with $m \geqslant 1$ and $n \geqslant 1$. The theorem will follow if we show that $|S| \geqslant \kappa_0(G)$ and $|S| \leqq \kappa_1'(G)$. To prove the first inequality, suppose that $|S| < \kappa_0(G)$. Let H be the graph obtained by removing all vertices of S from G. The edges of S then form a minimal separating edge set for H. Hence $m + n < \kappa_0(G) \leqq \kappa_0(H) + n \leqq \kappa_1(H) + n \leqq m + n$, a contradiction. To prove the second inequality, suppose $|S| > \kappa_1'(G)$. Since S is minimal and has at least one edge, $G - S$ has exactly two components. Furthermore, each edge in S must join the two components and each vertex in S must be adjacent to vertices in both. Consider the set of edges from vertices in S to just one of the components together with the edges in S. This set has at least $m + n$ edges and is a minimal separating edge set, a contradiction.

Separating sets containing both vertices and edges have previously been considered by Beineke and Harary [1], who define a *connectivity pair* for graph G as an ordered pair (k, l) of non-negative integers such that there is some set S having k vertices and l edges whose removal disconnects G and is minimal in the sense that no set having $k - 1$ vertices and l edges or having k vertices and $l - 1$ edges has this property. Such sets need not be proper subsets of S. Two connectivity pairs are $(\kappa_0(G), 0)$ and $(0, \kappa_1(G))$. An extension of Menger's theorem [4] is proved, namely that if (k, l) is a connectivity pair for vertices s and t in graph G, then there are $k + l$ edge disjoint paths joining s and t, of which k are mutually non-intersecting. For additional results on connectivity, see references [2], [3], and [5].

ACKNOWLEDGMENT

The authors wish to thank the referee for suggesting shorter proofs for Theorems 1 and 3.

REFERENCES

1. L. W. BEINEKE AND F. HARARY, The connectivity function of a graph, *Mathematika* **14** (1967), 197–202.
2. M. BEHZAD AND G. CHARTRAND, "Introduction to the Theory of Graphs," Allyn & Bacon Boston, 1972.
3. F. HARARY, "Graph Theory," Addison–Wesley, Reading, Mass., 1969.
4. K. MENGER, Zur allgemeinen Kurventheorie, *Fund. Math.* **10** (1927), 96–115.
5. W. TUTTE, "Connectivity in Graphs," Univ. of Toronto Press, Toronto, Canada, 1966.
6. H. WHITNEY, Congruent graphs and the connectivity of graphs, *Amer. J. Math.* **54** (1932), 150–168.